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Analytical Solution for Extensible Tethers

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THIS Note presents an analytical approximation to the solution for an *extensible tether* problem. The mathematical model considers a large particle (m_1), such as a space station, and a smaller particle (m_2), connected by an ideal, massless tether—one incapable of sustaining other than tensile loads.

For this solution, the main particle (m_1) is assumed to move on a circular orbit (r_1), at an angular rate $\dot{\phi}_1$. Both particles are constrained to a single plane of motion, with the smaller mass acted upon by the central mass (μ) gravitational attraction and the tether tension. The geometric description for this case is shown in Fig. 1.

Equations of Motion

Under the assumptions made of this problem: $F \equiv |\bar{F}_1| = |\bar{F}_2|$, $\dot{\phi}_1$ is the angular motion for the tethered mass system; the line of action for the tether tension is parallel to $\bar{l}$, and $\pm\theta$ locates the tether ($\bar{l}$) with respect to the *local* radial direction ($\bar{e}_x$). The local triad ($\bar{e}_x, \bar{e}_y, \bar{e}_z$) moves with the main particle, m_1. The mass m_1 moves at speed $V_1 (\equiv r_1 \dot{\phi}_1)$ in a direction parallel to $\bar{e}_y$, at each instant.

Based on the abovementioned assumptions and descriptions, the vector equation of motion for m_2, with respect to m_1, can be written as

$$\ddot{\bar{l}} = \dot{\phi}_1^2[(1-\Delta^{-3})\bar{r}_1 - \Delta^{-3}\bar{l}] - \frac{F\bar{l}}{\tilde{m}l} \tag{1}$$

Herein, because of the circular orbit, $\dot{\phi}_1^2 = \mu/r_1^3$, $\tilde{m} \equiv m_1 m_2/(m_1+m_2)$ is the reduced mass parameter. Also, the quantity

$$\Delta \equiv [1+2(l/r_1)\cos\theta + (l^2/r_1^2)]^{1/2} \tag{2}$$

is a parameter indicative of gravity gradient for this situation.

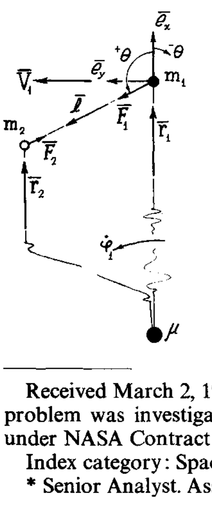

Fig. 1 Problem geometry. Unit vectors $\bar{e}_x$, $\bar{e}_y$ define the plane of motion.

Next, introducing the dimensionless quantities: $\lambda \equiv l/r_1$, $\tau \equiv F/\tilde{m}/r_1\dot{\phi}_1^2$, and, changing the independent variable from t to ϕ (i.e., $\phi \equiv \dot{\phi}_1 t$), then the scalar differential equations defining the problem are, from Eq. (1)

$$\lambda'' = \lambda(1+\theta')^2 + (1-\Delta^{-3})\cos\theta - \lambda\Delta^{-3} - \tau \tag{3a}$$

and

$$\lambda\theta'' = -2\lambda'(1+\theta') - (1-\Delta^{-3})\sin\theta$$

In these, the primes denote differentiation with respect to ϕ.

For this study the angle θ is set at a fixed value, for any particular solution; consequently Eqs. (3) reduce to the set

$$\lambda'' = (\lambda+\cos\theta)(1-\Delta^{-3}) - \tau \tag{4a}$$

and

$$\lambda' = -(1-\Delta^{-3})(\sin\theta/2) \tag{4b}$$

As an aid to obtaining the analytical solution desired here, since λ is a small term, we introduce the approximation

$$\Delta^{-3} \cong 1 - 3\lambda\cos\theta + \text{HOT}$$

correspondingly, Eqs. (4) are further reduced to

$$\lambda' \cong -\tfrac{3}{4}\lambda\sin 2\theta \tag{5a}$$

and

$$\tau \cong (3\lambda/2)(1+\cos 2\theta) - \lambda'' \tag{5b}$$

A more convenient "tether-length variable" can be had by introducing the dimensionless maximum value of $\lambda (\equiv \lambda_m)$. This leads to a new parameter replacing λ, namely

$$\sigma \equiv \lambda/\lambda_m = l/l_{\max} \tag{6a}$$

It is recognized that this new variable is constrained to

$$0 \leq \sigma \leq 1.0$$

during any tether *extension*, or *contraction*.

Next, for compatibility, define a new tension parameter, i.e., let

$$\tau_\sigma \equiv \frac{\tau}{\lambda_m} = \frac{F/\tilde{m}}{\dot{\phi}_1^2 l_{\max}} \tag{6b}$$

and rewrite Eqs. (5) accordingly.

The Solution

Remembering that θ is a fixed quantity, then Eq. (5a) leads directly to the first integral

$$\sigma = \sigma_o \exp(K_\sigma \phi) \tag{7}$$

wherein $K_\sigma \equiv -\frac{3}{4}\sin 2\theta$, and σ_o is the initial value of σ.

Making use of this result, then Eq. (5b) yields, as its "solution"

$$\tau_\sigma = [\tfrac{3}{2}(1+\cos 2\theta) - K_\sigma^2]\sigma \equiv K_\tau \sigma \tag{8}$$

Next, recalling that $\phi \equiv \dot{\phi}t$, Eq. (7) can be manipulated to produce the time equation

$$t = \frac{1}{K_\sigma \dot{\phi}} \ln\frac{\sigma}{\sigma_o} = \frac{1}{K_\sigma \dot{\phi}} \ln\left(\frac{l}{l_o}\right) \tag{9}$$

Finally, rewriting Eq. (5a) in terms of σ, i.e.,

$$\sigma' = -\tfrac{3}{4}\sigma\sin 2\theta \equiv K_\theta \sigma \tag{10}$$

then Eqs. (7–10) describe *this* "extensible tethered bodies" problem, completely. Interestingly, one sees that the rate of "reeling-in" or "reeling-out" (σ') of the tether is linearly related to the length (σ); correspondingly, the required tensile load (for the tether) is also linearly related to the line length. However, contrary to this, the "time" required to reel-in or reel-out the line is logarithmic in the length.

Discussion

As an interesting observation, Eq. (10) indicates that the reeling-in of the tethered mass (m_2) can be accommodated *only* in the *first* and *third* θ-quadrants. Conversely, the line may be *reeled-out* in the second and fourth quadrants, only.

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Another interesting consequence of these results is that this system does not operate at $\theta = n\pi/2$ orientations. That is, tethers cannot be used to lower or raise a mass along the local vertical directions, *nor* orthogonal to them, with θ fixed in value.

It is also apparent here that these operations cannot originate with $l = 0$; there *is* need for a finite gravity on the suspended mass particles, if the system is to function (mathematically and physically).

It should be evident that these tethered mass operations are simply dependent on (*only*) the θ-orientation and the circular orbit altitude; here altitude is implied by the parameter $\dot\phi_1$.

To initiate either a reel-in *or* reel-out operation, an appropriate set of initial conditions are required. That is, for some initial tether length the mass (m_2) must be given an impulsive state (σ_o'), and the tether must (simultaneously) acquire a tension (τ_σ) of proper magnitude. [This tension (obviously) depends on the line length]. At the terminus of an operation there will be an end transient required if one desires to establish a null state.

To the degree of approximation used here, all of the dimensionless quantities ($\sigma, \sigma', \tau_\sigma$), describing these tethered body operations, are independent of the main body's orbit. It is only the *time* expression which exhibits a dependence on orbit altitude (and this through $\dot\phi_1$). Because of the nature of these various dependences, most of the problem's behavior (with θ) can be demonstrated by graphing $K_\sigma(\theta)$ and $K_\tau(\theta)$. However, to illustrate the dependence of t on θ, a specific situation must be assumed. In this regard consider a circular orbit where $\dot\phi_1 = 10^{-3}$ rad/sec (this corresponds to an altitude of 981.64 km). Also, assume the tether is extended (or contracted) between 10 and 10^4 m. For these conditions, a graphing of t (to complete the extensible operation), as a function of θ, can be provided.

The information described previously is shown on Fig. 2. For the graph, it should be recognized that K_σ is positive in the range $\pi/2 \leq \theta \leq \pi$, and $3\pi/2 \leq \theta \leq 2\pi$, only; however, K_τ is positive for all θ. Also, it should be recalled that in the angle range, $0 \leq \theta \leq \pi/2$ and $\pi \leq \theta \leq 3\pi/2$, the system is capable of reeling-in only; reeling-out of the tether will occur in the other quadrants.

From a study of the curves on this figure it is seen that these operations are least sensitive to parameter variations in the vicinity of $\theta = (2n+1)\pi/4$.

An Example

In order to provide some insight into the size of the "numbers" which describe these operations, consider the following example: a main body circular orbit with $\dot\phi_1 = 10^{-3}$ rad/sec, and a tethered unit mass (m_2) which is to be *lowered* from 10 to 10^4 m along a straight line (relative) path at $\theta = 135°$.

a) The initial values for this operation are:

(from $\sigma' = K_\sigma \sigma$): $\dot l_o \equiv K_\sigma \sigma_o l_f \dot\phi = K_\sigma l_o \dot\phi = 0.075$ m/sec

(from $\tau_\sigma = K_\tau \sigma$): $(F/\tilde m)_o \equiv K_\tau \sigma l_f \dot\phi^2 = K_\tau l_f \dot\phi^2 = 0.9375$ (10^{-5}) m/sec^2.

b) The terminal values for the extension are:

$\dot l_f = K_\sigma l_f \dot\phi = 0.75(10^4)10^{-3} = 7.5$ m/sec;

$(F/\tilde m)_f = K_\tau l_f \dot\phi^2 = 0.9375(10^4)10^{-6} = 0.9375(10^{-2})\,m/sec^2$;

and, the time required to complete the operation:

$t_f = (1/K_\sigma \dot\phi)\ln(\sigma_f/\sigma_o) = (10^3/0.75)\ln(10^3) = 9210.34$ sec

or 153.506 min. (This corresponds to 1.466 orbits of the main body, m_1).

Conclusions

The problem situation described previously offers a practical solution which should be quite useful in certain space flight operations,[2] especially for the transfer of cargo and personnel and for retrieval and rescue operations. The idea is simple in its application and does not appear to require sophisticated hardware. An even more appealing advantage is that it is infinitely reusable; i.e., it could be rewound and used over and over again.

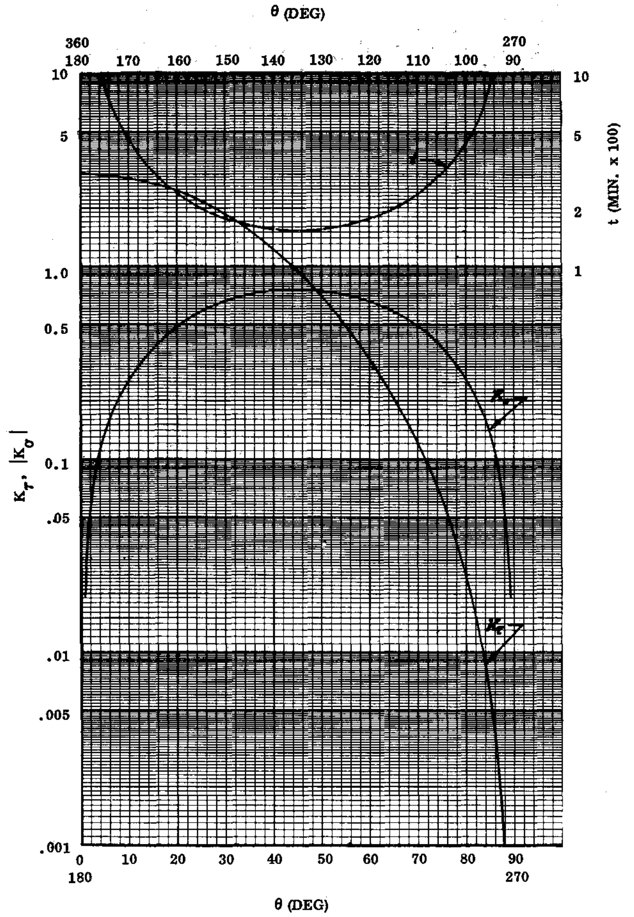

Fig. 2 Variations of K_σ and K_τ with θ. Also, the tether extension time is shown, for the sample situation noted, as a function of θ.

It is worth noting that this simple analysis has been compared to a more exact numerical study for a few sample cases. It was found that these approximate solutions agreed with the numerical results to better than 0.5%, for all comparisons.

There may be a question regarding how much θ varies during a maneuver if the full Eqs. (3) are used. For the sample cases studied, the numerical solution predicted a variation in θ of much less than a degree for the full "extension." A more descriptive discussion on this is found in Ref. 1, Sec. III, 3.9.

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Wobble Correction for a Dual-Spin Vehicle

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THE problem of bearing axis wobble in a dual-spin satellite has been analyzed by McIntyre and Gianelli.[1] They find that for a despun platform, symmetric in the sense that its

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